A model for pricing real estate derivatives with stochastic interest rates

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Abstract

The real estate derivatives market allows participants to manage risk and return from exposure to property, without buying or selling directly the underlying asset. Such market is growing very fast hence the need to rely on simple yet effective pricing models is very great. In order to take into account the real estate market sensitivity to the interest rate term structure in this paper is presented a two-factor model where the real estate asset value and the spot rate dynamics are jointly modeled. The pricing problem for both European and American options is then analyzed and since no closed-form solution can be found a bidimensional binomial lattice framework is adopted. The model proposed allows calibration to the interest rate and volatility term structures.

Key words: Real estate; derivatives pricing; stochastic interest rate; bidimensional binomial lattice.

1 Introduction

The derivatives pricing problem roots lie in the seminal papers by Black and Scholes [1] and Merton [2] (hereafter BSM). They were the first to analytically solve the option pricing problem and for this achievement in 1997 were awarded the Nobel prize. This paper relies on the BSM risk-neutral valuation framework adapting it to the peculiarity of the real estate derivatives. In fact this class of contracts is characterized by a payoff dependent on an underlying real estate asset whose value depends on the interest rate dynamics. In particular in this paper the real estate asset value is represented by a geometric Brownian

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motion whereas the spot interest rate, which in the standard BSM framework is considered as a constant parameter, is properly modeled as a stochastic variable.

The layout of the remainder of the paper is as follows. In section 2 we present the two-factor pricing model in continuous time and derive the general valuation equation for any real estate derivative depending on the real estate asset value and the spot interest rate. Since the level of analytical tractability of the continuous-time model is quite limited, a discrete-time version within a bidimensional binomial lattice framework is introduced. In section 3 we show how the bidimensional binomial lattice can be implemented by using state-contingent Arrow-Debreu prices and then calibrated to the current market term structure of interest rates and the current term structure of volatilities. In section 4 an application to European and American option pricing problems is given. Finally conclusions are drawn in section 5.

2 Model formulation

Let us consider an economy with two correlated state variables, the real estate asset value $X = \{X(t),\, t \geq 0\}$ and the risk-free spot interest rate $r = \{r(t),\, t \geq 0\}$, whose evolutions as function of the time variable t are described respectively by the following stochastic differential equations (SDEs)

$$dX(t) = \mu_X(X_t, t)dt + \sigma_X(X_t, t)dW_1(t),$$

$$dr(t) = \mu_r(r_t, t)dt + \sigma_r(r_t, t)dW_2(t),$$

where $\{W_i(t),\, t \geq 0\}$, for $i = 1, 2$, are two standard Wiener processes defined under the same natural probability measure $\mathbb{P}$ and correlated through a time-dependent correlation coefficient $\rho(t) \in [-1, 1]$, $\forall\, t \geq 0$.

In particular, for the real estate asset value X we specify a geometric Brownian motion

$$dX(t) = (\mu - \delta)X(t)dt \,+\, \sigma_X X(t)dW_1(t), \tag{2.1}$$

where $\delta \geq 0$ is the cash-flow continuously paid by the real estate asset and μ is its total expected rate of return, so that the difference $\mu - \delta$ is the real estate asset rate of appreciation per unit time. The coefficient $\sigma_X > 0$ indicates the instantaneous volatility of the real estate asset value.

The term structure is generated by a stochastic process modeling the dynamics of the natural logarithm of the spot interest rate r

$$d\ln r(t) = \left\{ \frac{\partial \ln u(t)}{\partial t} - \frac{\partial \ln \sigma_r(t)}{\partial t}\Big[\ln u(t) - \ln r(t)\Big]\right\} dt$$
$$+ \sigma_r(t)\, dW_2(t), \tag{2.2}$$

where $u(t)$ is the median of the spot rate distribution at time t and $\sigma_r(t)$ the spot rate volatility at time t. The spot rate process (2.2) is the continuous-time equivalent of the interest rate model developed by Black, Derman and Toy [3] (hereafter called BDT model).

Although the BDT model was originally presented in a discrete-time binomial lattice framework, equation (2.2) better clarifies the distinctive features of this arbitrage-free and yield-based model. In fact the two unknown time-dependent functions, $u(t)$ and $\sigma_r(t)$, are chosen to make the model consistent with, respectively, the current market term structure of interest rates (also known as the yield curve) and the current term structure of volatilities (also known as the volatility curve). A further advantage with respect to other term structure models is that interest rates cannot become negative, since changes in spot rates are lognormally distributed. Unfortunately, due to its lognormality, no analytic solution can be found and thus numerical techniques are required to derive an interest rate tree that correctly matches the market term structures. An uncomfortable consequence of the model is that for certain specification of the volatility function $\sigma_r(t)$ the spot interest rate can be mean-fleeing rather than mean-reverting. For this reason, many practitioners find it better to fit the model only to the interest rate term structure, holding the volatility term structure to a constant level σ_r. In this case the stochastic process (2.2) reduces to the following

$$d\ln r(t) = \frac{\partial \ln u(t)}{\partial t}\, dt \,+\, \sigma_r\, dW_2(t). \tag{2.3}$$

Applying Itô's lemma to (2.2) and assuming a risk-neutral valuation setting [1] the two-factor model can be written in matrix form as follows

$$\begin{bmatrix} dX(t) \\ \\ dr(t) \end{bmatrix} = \begin{bmatrix} \Big(r(t) - \delta\Big)X(t) \\ \\ \Big(\psi(t) + \frac{1}{2}\,\sigma_r(t)^2\Big)r(t) \end{bmatrix} dt \,+ \begin{bmatrix} \sigma_X X(t) & 0 \\ \\ 0 & \sigma_r(t)^2 r(t) \end{bmatrix} \begin{bmatrix} d\tilde{W}_1(t) \\ \\ d\tilde{W}_2(t) \end{bmatrix},$$

with

[1] For more details on risk-neutral valuation and real estate derivatives see [4] where is also applied a real estate derivative pricing model which, unlike the one presented in this paper, is not consistent with the current interest rate and volatility term structures.

$$\psi(t) = \frac{\partial \ln u(t)}{\partial t} - \frac{\partial \ln \sigma_r(t)}{\partial t}\Big[\ln u(t) - \ln r(t)\Big],$$

and where the terms $d\tilde{W}_1(t)$ and $d\tilde{W}_2(t)$ are increments of the two correlated standard Wiener processes defined now under the risk-neutral probability measure $\mathbb{Q}$.

Let $\Pi(t) = \Pi(X_t, r_t, t)$ be a continuous, twice-differentiable function of the state variables X and r at time t, and differentiable with respect to the time variable t. Applying the multivariate version of Itô's Lemma, we have

$$\begin{aligned} d\Pi(t) = & \left[\frac{\partial \Pi(t)}{\partial t} + \frac{\partial \Pi(t)}{\partial X}\Big(r(t) - \delta\Big)X(t) + \frac{\partial \Pi(t)}{\partial r}\Big(\psi(t) + \frac{1}{2}\,\sigma_r^2(t)\Big)r(t)\right. \\ & + \frac{1}{2}\,\frac{\partial^2 \Pi(t)}{\partial X^2}\,\sigma_X^2 X(t)^2 + \frac{1}{2}\,\frac{\partial^2 \Pi(t)}{\partial r^2}\,\sigma_r(t)^2\, r(t)^2 \\ & \left. + \frac{\partial^2 \Pi(t)}{\partial X \partial r}\,\rho(t)\,\sigma_X \sigma_r(t) X(t) r(t)\right] dt \\ & + \frac{\partial \Pi(t)}{\partial X}\,\sigma_X X(t) d\tilde{W}_1(t) + \frac{\partial \Pi(t)}{\partial r}\,\sigma_r(t)\, r(t) d\tilde{W}_2(t). \end{aligned}$$

Using the hedging argument yields the following second-order partial differential equation

$$\begin{aligned} & \frac{\partial \Pi(t)}{\partial t} + \frac{\partial \Pi(t)}{\partial X}\Big(r(t) - \delta\Big)X(t) + \frac{\partial \Pi(t)}{\partial r}\left[\Big(\psi(t) + \frac{1}{2}\,\sigma_r(t)^2\Big)\right. \\ & \left. + q(t)\sigma_r(t)\right] r(t) + \frac{1}{2}\,\frac{\partial^2 \Pi(t)}{\partial X^2}\,\sigma_X^2 X(t)^2 + \frac{1}{2}\,\frac{\partial^2 \Pi(t)}{\partial r^2}\,\sigma_r(t)^2\, r(t)^2 \\ & + \frac{\partial^2 \Pi(t)}{\partial X \partial r}\,\rho(t)\,\sigma_X \sigma_r(t) X(t) r(t) - r(t)\Pi(t) = 0, \end{aligned} \tag{2.4}$$

where $q(t)$ is the market price of the interest-rate risk. In order to determine the value Π of any contingent claim dependent upon X and r, we must solve numerically equation (2.4), subject to the appropriate terminal and boundary conditions.

For computational purposes it is convenient to approximate the joint evolution of the two continuous-time stochastic processes X and r with a bidimensional binomial (BB) lattice. In order to specify the jump sizes and probabilities we equate means, variances and correlations for the bidimensional binomial process with those of the stochastic processes X and r. Given that it is easier to work with an additive two-variable binomial process we consider equation (2.1) and derive the dynamics for the natural logarithm of the real estate asset value, i.e.

$$dy(t) = \nu(t)dt \,+\, \sigma_X d\tilde{W}_1(t), \tag{2.5}$$

where $y(t) := \ln X(t)$ and with the drift term $\nu(t)$ which is defined as

$$\nu(t) = \Big(r(t) - \delta - \frac{1}{2}\,\sigma_X^2\Big).$$

Let us assume that during the time period $[t,\, t+\Delta t]$ the natural logarithms of X and r can either go up to levels respectively of $y(t)+\Delta y_u(t)$ and $\ln r(t)+\Delta \ln r_u(t)$ or down to levels respectively of $y(t)+\Delta y_d(t)$ and $\ln r(t)+\Delta \ln r_d(t)$. Let $p_{uu}(t)$, $p_{ud}(t)$, $p_{du}(t)$ and $p_{dd}(t)$ denote the joint probabilities of the additive up and down jumps for the BB process, with the two subscripts representing the jump type, upward u or downward d, of y and $\ln r$, respectively. The time-dependent sets of additive jump sizes $\{\Delta y_u(t),\, \Delta y_d(t),\, \Delta \ln r_u(t),\, \Delta \ln r_d(t)\}$ and joint probabilities $\{p_{uu}(t),\, p_{ud}(t),\, p_{du}(t),\, p_{dd}(t)\}$ are chosen to match the first and the second moments of the risk-neutral processes, i.e.,

$$\begin{aligned}\mathbb{E}\Big(\Delta y(t)\Big) &:= \Big(p_{uu}(t) + p_{ud}(t)\Big)\Delta y_u(t) + \Big(p_{du}(t) + p_{dd}(t)\Big)\Delta y_d(t)\\ &= \nu(t)\Delta t,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\mathbb{E}\Big(\Delta y(t)^2\Big) &:= \Big(p_{uu}(t) + p_{ud}(t)\Big)\Delta y_u(t)^2 + \Big(p_{du}(t) + p_{dd}(t)\Big)\Delta y_d(t)^2\\ &= \sigma_X^2\Delta t + \nu(t)^2\Delta t^2,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\mathbb{E}\Big(\Delta \ln r(t)\Big) &:= \Big(p_{uu}(t) + p_{du}(t)\Big)\Delta \ln r_u(t) + \Big(p_{ud}(t) + p_{dd}(t)\Big)\Delta \ln r_d(t)\\ &= \psi(t)\Delta t,\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathbb{E}\Big(\Delta \ln r(t)^2\Big) &:= \Big(p_{uu}(t) + p_{du}(t)\Big)\Delta \ln r_u^2 + \Big(p_{ud}(t) + p_{dd}(t)\Big)\Delta \ln r_d^2\\ &= \sigma_r(t)^2\Delta t + \psi(t)^2\Delta t^2,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathbb{E}\Big(\Delta y(t)\Delta \ln r(t)\Big) &:= \Big(p_{uu}(t) - p_{du}(t)\Big)\Delta y(t)\Delta \ln r_u(t)\\ &\quad + \Big(p_{ud}(t) - p_{dd}(t)\Big)\Delta y(t)\Delta \ln r_d(t)\\ &= \rho(t)\,\sigma_X\sigma_r(t)\Delta t + \nu(t)\,\psi(t)\Delta t^2,\end{aligned} \tag{2.10}$$

where the joint probabilities must satisfy the following constraint

$$p_{uu}(t) + p_{ud}(t) + p_{du}(t) + p_{dd}(t) = 1. \tag{2.11}$$

Note that the system of equations listed above cannot be solved analytically since there are more unknowns than constraints. In order to ensure the analytical tractability of the model we set the upward and downward jump sizes for $y(t)$ to be equal, that is

$$\Delta y_u(t) = -\Delta y_d(t) \quad \text{with} \quad \Delta y_u(t) \equiv \Delta y(t). \tag{2.12}$$

The above condition is similar to that originally proposed by [5] and on average has slightly better accuracy than the standard univariate binomial model with equal probabilities of one-half developed by [6]. Finally, we impose that the unconditional probabilities of upward and downward jumps for the short-rate dynamics are both equal to one-half. This choice in conjunction with equation (2.11) leads to the following constraints

$$p_{uu}(t) + p_{du}(t) := p_r = \frac{1}{2}, \tag{2.13}$$

$$p_{ud}(t) + p_{dd}(t) = 1 - p_r := q_r = \frac{1}{2}. \tag{2.14}$$

Since the system consisting of equations (2.6)-(2.10) and (2.12)-(2.14) can now be solved analytically, we obtain for the additive jump sizes, $\Delta y(t)$, $\Delta \ln r_u(t)$ and $\Delta \ln r_d(t)$, and the joint probabilities, $p_{uu}(t)$, $p_{ud}(t)$, $p_{du}(t)$, and $p_{dd}(t)$, the following expressions in terms of model parameters

$$\Delta y(t) = \sqrt{\sigma_X^2 \Delta t + \nu(t)^2 \Delta t^2}, \tag{2.15}$$

$$\Delta \ln r_u(t) = \psi(t)\Delta t + \sigma_r(t)\sqrt{\Delta t}, \tag{2.16}$$

$$\Delta \ln r_d(t) = \psi(t)\Delta t - \sigma_r(t)\sqrt{\Delta t}, \tag{2.17}$$

and

$$p_{uu}(t) = \frac{1}{4}\left(1 + \frac{\nu(t)\sqrt{\Delta t}}{\sqrt{\sigma_X^2 + \nu(t)^2\Delta t}} + \frac{\rho(t)\,\sigma_X}{\sqrt{\sigma_X^2 + \nu(t)^2\Delta t}}\right), \tag{2.18}$$

$$p_{ud}(t) = \frac{1}{4}\left(1 + \frac{\nu(t)\sqrt{\Delta t}}{\sqrt{\sigma_X^2 + \nu(t)^2\Delta t}} - \frac{\rho(t)\,\sigma_X}{\sqrt{\sigma_X^2 + \nu(t)^2\Delta t}}\right), \tag{2.19}$$

$$p_{du}(t) = \frac{1}{4}\left(1 - \frac{\nu(t)\sqrt{\Delta t}}{\sqrt{\sigma_X^2 + \nu(t)^2\Delta t}} - \frac{\rho(t)\,\sigma_X}{\sqrt{\sigma_X^2 + \nu(t)^2\Delta t}}\right), \tag{2.20}$$

$$p_{dd}(t) = \frac{1}{4}\left(1 - \frac{\nu(t)\sqrt{\Delta t}}{\sqrt{\sigma_X^2 + \nu(t)^2\Delta t}} + \frac{\rho(t)\,\sigma_X}{\sqrt{\sigma_X^2 + \nu(t)^2\Delta t}}\right). \tag{2.21}$$

In order that each probability lies within the interval $[0, 1]$ it must be satisfied the following condition

$$\frac{\left|\nu(t)\right|\sqrt{\Delta t}\,+\,\left|\rho(t)\right|\sigma_X}{\sqrt{\sigma_X^2+\nu(t)^2\Delta t}}\leq 1. \tag{2.22}$$

Solving (2.22) with respect to the correlation coefficient $\rho(t)$ yields

$$\left|\rho(t)\right|\leq\frac{\sqrt{\sigma_X^2+\nu(t)^2\Delta t}\,-\,\left|\nu(t)\right|\sqrt{\Delta t}}{\sigma_X},\qquad\forall\, t\geq 0\,. \tag{2.23}$$

Therefore within of our pricing model an almost arbitrary degree of correlation satisfying inequalities (2.23) can be accommodated between the real estate asset value and the spot interest rate. By virtue of its Markovian nature, this arbitrage-free two-factor model can be mapped onto a recombining BB tree, and therefore readily lends itself to the evaluation of a wide class of interest rate sensitive derivative securities.

3 Implementation and calibration of the bidimensional binomial tree

In this section we show how the BB lattice can be built to represent the dynamics of the two correlated state variables. Firstly, the BB lattice is implemented in such a way that it approximates the SDEs for the real estate asset value and the spot interest rate and then calibrated so that it is consistent with the current market term structures.

Let us assume that the calibrated bivariate binomial lattice has N periods and each period is of size Δt years. Hence the total time horizon of the lattice is $T = N\Delta t$ years. The recombining nature of the BB lattice ensures that at a generic time step $n = 0, 1, 2, \ldots, N$, corresponding to time $t = n\Delta t$, there are $(n+1)^2$ nodes which we label as (n, i, j), with $i = -n, -n+2, \ldots, n-2, n$ and $j = -n, -n+2, \ldots, n-2, n$ representing the levels achieved respectively by the state variables X and r. Hence at each period n and for both variables, the possible states are separated by two space steps and space indices i and j, with $i, j \in \mathbb{Z}$, will step by two. It is useful to divide the $(n + 1)^2$ nodes at every period n into the following three categories:

a) four *extreme nodes*, corresponding to the four possible combinations of the extreme levels of the two state variables. These nodes, which are referred to by (n, i, j), with $i = \pm n$ and $j = \pm n$, can be reached by a unique transitional path;
b) $[(n + 1) - 2] \times 4$ *external nodes*, corresponding to all of the possible combinations of the extreme levels for one of the two state variables and of the intermediate levels for the other. These nodes, which are referred

to by (n,i,j), with either $i=\pm n$ and $|j|\leq n-2$ or $|i|\leq n-2$ and $j=\pm n$, can be reached by two transitional paths;

c) $[(n+1)-2]^2$ *internal nodes*, corresponding to all the possible combinations of the intermediate levels for both state variables. These nodes, which are referred to by (n,i,j), with $|i|\leq n-2$ and $|j|\leq n-2$, can be reached by four transitional paths.

Figure 1 illustrates the nodes and the transitional paths in a BB tree with $N=2$ periods.

[Figure 1 about here]

Let $X(n,i,j)$ and $r(n,j)$ be, respectively, the real estate asset value and the (annualized) one-period short rate at node (n,i,j) of the bidimensional binomial lattice. Furthermore, let $p_{uu}(n,j)$, $p_{ud}(n,j)$, $p_{du}(n,j)$ and $p_{dd}(n,j)$ denote the joint probabilities of the up and down movements for the BB tree at node (n,i,j) and as functions of the corresponding short rate $r(n,j)$.

As it is shown in Figure 2, there are four branches emerging from the node (n,i,j) to represent the four possible combinations of the two state variables going up or down at time step $n+1$. While the short (Δt-period) rate $r(n,j)$ can evolves either to a down-state, i.e.

$$r(n+1,j-1), \qquad \text{with probability } p_{ud}(n,j)+p_{dd}(n,j)=q_r,$$

or to an up-state, i.e.

$$r(n+1,j+1), \qquad \text{with probability } p_{uu}(n,j)+p_{du}(n,j)=p_r,$$

the real estate asset value $X(n,i,j)$ rises or falls by an amount conditional on the interest rate movements as follows

$$\begin{cases} X(n+1,i+1,j+1), & \text{with probability } p_{uu}(n,j) \\ X(n+1,i+1,j-1), & \text{with probability } p_{ud}(n,j) \\ X(n+1,i-1,j+1), & \text{with probability } p_{du}(n,j) \\ X(n+1,i-1,j-1), & \text{with probability } p_{dd}(n,j) \end{cases}$$

Hence $X(n+1,i+1,j-1)$ and $X(n+1,i-1,j-1)$ represent up and down values conditional on a downward movement of the short rate, while $X(n+1,i+1,j+1)$ and $X(n+1,i-1,j+1)$ are up and down values if otherwise an upward movement of the short rate occurs. Note that the joint probabilities, $p_{uu}(n,j)$, $p_{ud}(n,j)$, $p_{du}(n,j)$, and $p_{dd}(n,j)$, are defined by equations (2.18)-(2.21) with the drift term $\nu(t)$ approximated now by

$$\nu(n,j) = \left(r(n,j) - \delta - \frac{1}{2}\sigma_X^2\right).$$

The recombining BB lattice can be constructed efficiently, extending the technique of forward induction first introduced by [7]. The procedure is an implementation of the binomial formulation of the Fokker-Planck forward equation and it is applicable to the general class of term structure models which is referred to as "Brownian-path independent" and that includes, among others, the BDT model.

[Figure 2 about here]

Following [7] in our two-factor pricing model the levels of the two state variables at time t, i.e. the natural logarithm of the real estate asset value $y(t)$ and the spot interest rate $r(t)$, are given respectively by

$$y(t) = U_X(t) + \sigma_X \tilde{W}_1(t), \tag{3.1}$$

$$r(t) = U_r(t) \exp\left(\sigma_r(t)\tilde{W}_2(t)\right), \tag{3.2}$$

where $U_X(t)$ is the mean of the normal distribution for y at time t, $U_r(t)$ is the median of the lognormal distribution for r at time t, σ_X and $\sigma_r(t)$ are the levels (in percentage terms) of the constant and time-dependent volatilities of X and r respectively, $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ are the levels of the two correlated standard Wiener processes defined under the same risk-neutral probability measure $\mathbb{Q}$. While the term $U_X(t)$ is a known function of the time variable t, that is

$$U_X(t) = y(0) + \nu(t)t, \tag{3.3}$$

where $y(0) := \ln X(0)$, with $X(0)$ the initial value of the real estate asset, the two unknown time-dependent functions $U_r(t)$ and $\sigma_r(t)$ must be determined at each time period in order to fit the model to the current market term structures. If the model is implemented to fit just the interest rate term structure, with $\sigma_r(t)$ set equal to a constant level σ_r, we only have to determine the median $U_r(t)$ and thus the level of the spot interest rate is given by

$$r(t) = U_r(t) \exp\left(\sigma_r \tilde{W}_2(t)\right). \tag{3.4}$$

Since as $\Delta t \to 0$ the BB process $\left(i\sqrt{\Delta t}, j\sqrt{\Delta t}\right)$ converges to the bidimensional standard Wiener process $\left(\tilde{W}_1(t),\, \tilde{W}_1(t)\right)$ we can represent the real estate asset value and the short rate levels in the BB lattice respectively as

$$X(n,i,j) = X(0)\exp\Big(i\Delta y(n,j)\Big), \tag{3.5}$$

$$r(n,j) = U_r(n)\exp\Big(\sigma_r(n)j\sqrt{\Delta t}\Big), \tag{3.6}$$

with $|i| \leq n$ and $|j| \leq n$ and where the equal jump size $\Delta y(n,j)$ for the natural logarithm of the real estate asset value is defined by (2.15). To build the recombining tree for the two state variables (i.e. determining $X(n,i,j)$ and $r(n,j)$ for each time period n and levels i and j) therefore requires us to determine $U_r(n)$ and $\sigma_r(n)$.

3.1 Determining the time-dependent functions $U_r(t)$ and $\sigma_r(t)$

In order to determine the time-dependent functions $U_r(t)$ and $\sigma_r(t)$ we resort to the forward induction technique which involves using the Arrow-Debreu securities. Let us assume to have a security which pays the following monetary units

$$\begin{cases} 1, & \text{if node } (n,i,j) \text{ is reached} \\ 0, & \text{otherwise} \end{cases}$$

and let $A(n,i,j)$ denote the value at time 0 (i.e., at root node $(0,0,0)$) of this Arrow-Debreu security that represents the building block of any security. In particular, the price of a pure discount bond which matures at period $n+1$ can be written in terms of the Arrow-Debreu prices as follows

$$B(n+1) = \sum_i \sum_j A(n,i,j)d(n,j), \tag{3.7}$$

with the two summations that take place across all of the possible nodes at period n, that is for $|i| \leq n$ and $|j| \leq n$, and where $d(n,j)$, which denotes the price at period n and state j of the zero-coupon bond maturing at period $n+1$ (i.e. the one-period discount factor at nodes (n,i,j), for all $|i| \leq n$), can be defined as follows

$$d(n,j) = \begin{cases} \dfrac{1}{1+r(n,j)\Delta t}, & \text{for simple compounding} \\ \exp\Big[-r(n,j)\Delta t\Big], & \text{for continuous compounding} \end{cases} \tag{3.8}$$

As pointed out by [8], the stability of lognormal short rate models is ensured by using the simple or effective annual rates instead of the continuously compounding interest rates.

The forward induction procedure involves accumulating the state-contingent prices as we progress through the tree. Specifically, the Arrow-Debreu prices at period n, level i for the variable X and level j for the variabile r (i.e., the $A(n, i, j)$'s), can be computed from the known values at period $n-1$ taking into account all possibile transitional paths that lead into node (n, i, j).

Firstly, each of the four extreme nodes at period n, with $n = 1, 2 \ldots, N$, can be reached by a unique path and thus the Arrow-Debreu prices satisfy the following recursive relation

$$A(n,i,j) = \begin{cases} p_{ud}(n-1,j+1)\,A(n-1,i-1,j+1)\,d(n-1,j+1), \\ \qquad\qquad\qquad\qquad i=n,\ j=-n \\ p_{uu}(n-1,j-1)\,A(n-1,i-1,j-1)\,d(n-1,j-1), \\ \qquad\qquad\qquad\qquad i=n,\ j=n \\ p_{dd}(n-1,j+1)\,A(n-1,i+1,j+1)\,d(n-1,j+1), \\ \qquad\qquad\qquad\qquad i=-n,\ j=-n \\ p_{du}(n-1,j-1)\,A(n-1,i+1,j-1)\,d(n-1,j-1), \\ \qquad\qquad\qquad\qquad i=-n,\ j=n \end{cases} \tag{3.9}$$

that is, we multiply each transitional probability by its state-contingent price at previous period $n-1$ and the corresponding one-period discount factor.

For each of the $[(n+1)-2] \times 4$ external nodes there are two transitional paths to be considered and therefore the Arrow-Debreu prices for such nodes at period n are computed recursively according to the following equation

$$A(n,i,j) = \begin{cases} p_{uu}(n-1,j-1)\,A(n-1,i-1,j-1)\,d(n-1,j-1) \\ \quad + p_{ud}(n-1,j+1)\,A(n-1,i-1,j+1)\,d(n-1,j+1), \\ \qquad\qquad\qquad\qquad i=n,\ |j| \le n-2 \\ p_{dd}(n-1,j+1)\,A(n-1,i+1,j+1)\,d(n-1,j+1) \\ \quad + p_{du}(n-1,j-1)\,A(n-1,i+1,j-1)\,d(n-1,j-1), \\ \qquad\qquad\qquad\qquad i=-n,\ |j| \le n-2 \\ p_{dd}(n-1,j+1)\,A(n-1,i+1,j+1)\,d(n-1,j+1) \\ \quad + p_{ud}(n-1,j+1)\,A(n-1,i-1,j+1)\,d(n-1,j+1), \\ \qquad\qquad\qquad\qquad |i| \le n-2,\ j=-n \\ p_{uu}(n-1,j-1)\,A(n-1,i-1,j-1)\,d(n-1,j-1) \\ \quad + p_{du}(n-1,j-1)\,A(n-1,i+1,j-1)\,d(n-1,j-1), \\ \qquad\qquad\qquad\qquad |i| \le n-2,\ j=n \end{cases} \tag{3.10}$$

that is, for each pair of nodes that lead into (n,i,j) we sum the two state-contingent prices at previous period $n-1$, multiplied by their corresponding probabilities and one-period discount factors.

Finally, for each of the $[(n+1)-2]^2$ internal nodes, the Arrow-Debreu prices at period n satisfy the following equation

$$\begin{aligned} A(n,i,j) = & \, p_{du}(n-1,j-1)\, A(n-1,i+1,j-1)\, d(n-1,j-1) \\ & + p_{dd}(n-1,j+1)\, A(n-1,i+1,j+1)\, d(n-1,j+1) \\ & + p_{uu}(n-1,j-1)\, A(n-1,i-1,j-1)\, d(n-1,j-1) \\ & + p_{ud}(n-1,j+1)\, A(n-1,i-1,j+1)\, d(n-1,j+1), \\ & \qquad |i| \leq n-2,\ |j| \leq n-2, \end{aligned} \tag{3.11}$$

that shows the four possibile transitional paths through which the internal node (n,i,j) can be reached moving on the BB lattice from the previous period $n-1$. Note that the Arrow-Debreu price at the initial period $n=0$ and level 0 for both state variables, i.e., the initial condition for the recursive procedure, is by definition given by $A(0,0,0)=1$.

Before describing the general procedure to implement and calibrate the BB lattice so that it is consistent with the current interest rate and volatility term structures, in the next section we show how a version of the BB tree fitted to the yield curve only can be efficiently constructed.

3.2 Fitting the yield curve only

Many practitioners when using the BDT model, set the volatility function $\sigma_r(t)$ to a constant level σ_r and so only fit to the yield curve. It follows that the mean-reverting term within the drift function $\psi(t)$ equates to zero, and then the spot rate process is described by the SDE (2.3) whereas its discrete-time representation is given by

$$r(n,j) = U_r(n) \exp\left(\sigma_r j \sqrt{\Delta t}\right), \qquad |j| \leq n. \tag{3.12}$$

Using equations (3.12) and (3.7) and simple compounding for the one-period discount factor $d(n,j)$ as expressed by equation (3.8), the price of the pure discount bond maturing at period $n+1$ can be rewritten as

$$B(n+1) = \sum_i \sum_j A(n,i,j) \frac{1}{1 + U_r(n) \exp\left(\sigma_r j \sqrt{\Delta t}\right) \Delta t}. \tag{3.13}$$

Given that the discount function $d(n, j)$ is determined by the market data, the only unknown in equation (3.13) is the median $U_r(n)$ of the lognormal distribution for r at period n. Due to the analytical intractability of the lognormal models, we cannot rearrange equation (3.13) to obtain $U_r(n)$ explicitly, and then we need to use a suitable numerical search technique. In order to solve numerically equation (3.13) and then get an approximated value for $U_r(n)$, by which to determine the short rate at period n using equation (3.12), we resort to the Newton-Raphson method.

Let $n \geq 1$ and assume that $U_r(n-1)$, $A(n-1, i, j)$, $r(n-1, j)$, $d(n-1, j)$ and $\{p_{uu}(n-1, j),\ p_{ud}(n-1, j),\ p_{du}(n-1, j),\ p_{dd}(n-1, j)\}$ have been found for all states i and j at period $n-1$. The values at the initial time, i.e. period $n = 0$, are $U_r(0) = r(0, 0) = Y(1)$, $A(0, 0, 0) = 1$, $d(0, 0) = 1/(1 + r(0, 0)\,\Delta t)$. The procedure can be summarized by the following steps:

Step 1: From the initial yield curve, compute the market price of the n-period pure discount bond $\hat{B}(n)$, for $n = 1, 2 \ldots, N+1$;
Step 2: Using recursive forward equations (3.9)–(3.11) relative to the three types of nodes, generate the Arrow-Debreu prices $A(n, i, j)$, for $|i| \leq n$ and $|j| \leq n$, with $n \leq N$;
Step 3: For $n \leq N$, substitute $\hat{B}(n+1)$ into nonlinear equation (3.13) and solve it for the only unknown $U_r(n)$ by using Newton-Raphson method;
Step 4: From $U_r(n)$ calculate $r(n, j)$ and $d(n, j)$, for $|j| \leq n$, with $n \leq N$, using equations (3.12) and (3.8), respectively;
Step 5: From $r(n, j)$, with $|j| \leq n$, compute the joint probabilities $p_{uu}(n, j)$, $p_{ud}(n, j)$, $p_{du}(n, j)$ and $p_{dd}(n, j)$ by equations (2.18)–(2.21) and calculate $X(n, i, j)$ by equation (3.5), for $|i| \leq n$, with $n \leq N$.

3.3 Fitting interest rate and volatility term structures

In this section we present the implementation procedure of the BB tree in its full generality. In order to fit the tree to both interest rate and volatility term structures we must consider the discrete-time equivalent of the spot rate process as expressed in equation (3.6).

Let $B_{UU}(n)$, $B_{DU}(n)$, $B_{UD}(n)$, and $B_{DD}(n)$ denote the four possibile prices at period 1 of a pure discount bond maturing at period n, with $n = 1, 2, \ldots, N+1$, and $Y_{UU}(n)$, $Y_{DU}(n)$, $Y_{UD}(n)$, and $Y_{DD}(n)$ be the corresponding yields. Given that at period 1 there are only two different realizations for the short (Δt-period) rate, $r(1, -1)$ and $r(1, 1)$, it implies that $B_{UU}(n) = B_{DU}(n) \equiv B_U(n)$ and $B_{UD}(n) = B_{DD}(n) \equiv B_D(n)$. Consequently, we have only two possibile yields, $Y_U(n) \equiv Y_{UU}(n) = Y_{DU}(n)$ and $Y_D(n) \equiv Y_{UD}(n) = Y_{DD}(n)$. Since upward and downward moves of the short rate differ by the factor $\exp\left[2\sigma_Y(n)\sqrt{\Delta t}\,\right]$ it follows that

$$\frac{Y_U(n)}{Y_D(n)} = \exp\Big[2\sigma_Y(n)\sqrt{\Delta t}\,\Big]. \tag{3.14}$$

where $\sigma_Y(n)$ denote the initial volatility corresponding to the yield on a pure discount bond which matures at period $n \geq 1$. Recalling the two possible ways to define the relationship between the initial price $B(0,n) \equiv B(n)$ and the corresponding yield $Y(0,n) \equiv Y(n)$ of a n-maturity pure discount bond, that is

$$B(n) = \begin{cases} \dfrac{1}{\Big[1+Y(n)\Delta t\Big]^n}, & \text{for simple compounding} \\ \\ \exp\Big[-Y(n)n\Delta t\Big], & \text{for continuous compounding} \end{cases} \tag{3.15}$$

we can solve for the initial yield volatility $\sigma_Y(n)$ in equations (3.14) to obtain the following formula

$$\sigma_Y(n) = \frac{1}{2\sqrt{\Delta t}} \ln\left(\frac{Y_U(n)}{Y_D(n)}\right). \tag{3.16}$$

The two different discount functions $B_U(n)$ and $B_D(n)$, for $n \geq 2$, are in relation to the prices at the initial time of n-maturity pure discount bonds, $B(n)$, according the following discounted expectation formula

$$B(n) = \frac{1}{1+r(0,0)\Delta t}\Big[B_U(n)\Big(p_{uu}(0,0)+p_{du}(0,0)\Big) \\ + B_D(n)\Big(p_{ud}(0,0)+p_{dd}(0,0)\Big)\Big]. \tag{3.17}$$

Using the probability conditions (2.13)–(2.14) and the continuous compounding as expressed in equation (3.15), we solve simultaneously equations (3.16) and (3.17) and find the following system of nonlinear equations

$$\begin{cases} B_D(n) = B_U(n)^{\exp\left[-2\sigma_Y(n)\sqrt{\Delta t}\right]} \\ \\ B_U(n) + B_U(n)^{\exp\left[-2\sigma_Y(n)\sqrt{\Delta t}\right]} = 2B(n)\Big[1+r(0,0)\Delta t\Big] \end{cases} \tag{3.18}$$

In order to determine the time-dependent functions that match the yield and volatility curves we use the forward induction technique which now involves defining the Arrow-Debreu securities as seen from the four possibile nodes at period 1. The following notation is then required:

$A_{UU}(n,i,j)$: Arrow-Debreu price at node $(1,1,1)$ of a security that pays off 1 if states i and j are realized at period n and 0 otherwise;

$A_{DU}(n,i,j)$: Arrow-Debreu price at node $(1,-1,1)$ of a security that pays off 1 if states i and j are realized at period n and 0 otherwise;
$A_{UD}(n,i,j)$: Arrow-Debreu price at node $(1,1,-1)$ of a security that pays off 1 if states i and j are realized at period n and 0 otherwise;
$A_{DD}(n,i,j)$: Arrow-Debreu price at node $(1,-1,-1)$ of a security that pays off 1 if states i and j are realized at period n and 0 otherwise.

Note that $A_{UU}(n,i,j) = A_{DU}(n,i-2,j)$ and $A_{UD}(n,i,j) = A_{DD}(n,i-2,j)$, for $|i| \leq n$. It follows that the initial condition for the recursive condition is then given by $A_{UU}(1,1,1) = A_{UU}(1,-1,1) = 1$ and $A_{UD}(1,1,1) = A_{DD}(1,-1,1) = 1$. Therefore, the prices at period 1 of $(n+1)$-maturity pure discount bonds, i.e. $B_U(n+1)$ and $B_D(n+1)$, for $n = 1, 2, \ldots, N$, can be written in terms of the newly defined Arrow-Debreu prices in the two equivalent forms as follows

$$B_U(n+1) \equiv \begin{cases} B_{UU}(n+1) = \sum_i \sum_j A_{UU}(n,i,j)d(n,j), \\ \qquad\qquad \text{at node } (1,1,1) \\ B_{DU}(n+1) = \sum_i \sum_j A_{DU}(n,i,j)d(n,j), \\ \qquad\qquad \text{at node } (1,-1,1) \end{cases} \tag{3.19}$$

$$B_D(n+1) \equiv \begin{cases} B_{UD}(n+1) = \sum_i \sum_j A_{UD}(n,i,j)d(n,j), \\ \qquad\qquad \text{at node } (1,1,-1) \\ B_{DD}(n+1) = \sum_i \sum_j A_{DD}(n,i,j)d(n,j), \\ \qquad\qquad \text{at node } (1,-1,-1) \end{cases} \tag{3.20}$$

where the one-period discount factor, $d(n,j)$, is defined using the simple compounding formula given in equation (3.8), that is

$$d(n,j) = \frac{1}{1 + r(n,j)\Delta t} = \frac{1}{1 + U_r(n)\exp\left(\sigma_r(n) j \sqrt{\Delta t}\right)\Delta t}.$$

Given that the term structure of pure discount bond prices and the term structure of yield volatilities, i.e. $\hat{B}(n)$ and $\hat{\sigma}_Y(n)$ for $n \geq 1$, are known at the initial time from market data, we are able to find the two discount functions, $B_U(n)$ and $B_D(n)$ for each period $n \geq 1$, by using the system of nonlinear equations (3.18) in conjunction with the Arrow-Debreu pricing formulae (3.19)–(3.20) in a bidimensional Newton-Raphson iteration scheme where there are two unknowns, $U_r(n)$ and $\sigma_r(n)$, to be solved simultaneously. Note that the prices $A_{UU}(n,i,j)$, $A_{DU}(n,i,j)$, $A_{UD}(n,i,j)$ and $A_{DD}(n,i,j)$ of the newly defined Arrow-Debreu securities are updated according to the three types of node (i.e. extreme, external or internal) using recursive relations analogous to equations (3.9)–(3.11).

Let $n \geq 1$ and assume that $U_r(n-1)$, $\sigma_r(n-1)$, $A_{UU}(n-1,i,j)$, $A_{DU}(n-1,i,j)$, $A_{UD}(n-1,i,j)$ $A_{DD}(n-1,i,j)$, $r(n-1,j)$, $d(n-1,j)$ and $\{p_{uu}(n-1,j)$, $p_{ud}(n-1,j)$, $p_{du}(n-1,j)$, $p_{dd}(n-1,j)\}$ have been found for all states i and j at period $n-1$. The values at the initial time are $U_r(0) = r(0,0) = Y(1)$, $A_{UU}(1,1,1) = A_{DU}(1,-1,1) = 1$, $A_{UD}(1,1,-1) = A_{DD}(1,-1,-1) = 1$, $\sigma_r(0) = \sigma_Y(1)$ and $d(0,0) = 1/(1 + r(0,0)\,\Delta t)$. The procedure consists of the following steps:

Step 1: From the initial yield and volatility curves, compute the market price and the corresponding yield volatility of the n-period pure discount bond, i.e. $\hat{B}(n)$ and $\hat{\sigma}_Y(n)$, for each period $n \geq 1$;

Step 2: Substitute $\hat{B}(n)$ and $\hat{\sigma}_Y(n)$ into system of nonlinear equation (3.18) to derive $B_U(n)$ and consequently $B_D(n)$, for $n \geq 2$, by using Newton-Raphson iteration technique;

Step 3: Using recursive forward relations analogous to equations (3.9)–(3.11) for three types of nodes, generate the Arrow-Debreu prices $A_{UU}(n,i,j)$, $A_{DU}(n,i,j)$, $A_{UD}(n,i,j)$ and $A_{DD}(n,i,j)$, for $|i| \leq n$ and $|j| \leq n$;

Step 4: Substitute $B_U(n+1)$ and $B_D(n+1)$ into nonlinear equations (3.19)–(3.20) and solve them for the two unknowns $U_r(n)$ and $\sigma_r(n)$ by using bidimensional Newton-Raphson method;

Step 5: From $U_r(n)$ and $\sigma_r(n)$ calculate $r(n,j)$ and $d(n,j)$, for $|j| \leq n$, using equations (3.6) and (3.8), respectively;

Step 6: From $r(n,j)$, with $|j| \leq n$, compute the probabilities $p_{uu}(n,j)$, $p_{ud}(n,j)$, $p_{du}(n,j)$ and $p_{dd}(n,j)$ by equations (2.18)–(2.21) and calculate $X(n,i,j)$ by equation (3.5), for $|i| \leq n$.

4 An application to European and American options

An attractive feature of the BB lattice framework presented in this paper lies in the fact that, once the tree is built, any security dependent upon the two state variables can be easily evaluated by backward induction.

Let $\Pi(n,i,j)$ be the value of a real estate derivative at time step $n < N$, at level i in the underlying real estate asset value and at level j in the short rate. Within our two-factor pricing model, the value $\Pi(n,i,j)$ of the derivative security is obtained as the discounted present value of the four possible future prices at time step $n+1$ by the following backward equation

$$
\begin{aligned}
\Pi(n,i,j) = d(n,j)\Big[&\Pi(n+1,i+1,j-1)p_{ud}(n,j)\\
&+\Pi(n+1,i-1,j-1)p_{dd}(n,j)\\
&+\Pi(n+1,i+1,j+1)p_{uu}(n,j)\\
&+\Pi(n+1,i-1,j+1)p_{du}(n,j)\Big],
\end{aligned}
\tag{4.1}
$$

where the one-period discount factor $d(n,j)$ is defined by (3.8). This iteration continues backward all the way to the initial time $n=0$. The initial value of the derivative security is then given by $\Pi_0 \equiv \Pi(0,0,0)$.

The pricing problem of any contingent claim dependent upon the two state variables can be solved using the same general backward iteration procedure, but the distinctive features that characterize a specific derivative contract are entirely embodied in the terminal and boundary conditions that, therefore, must be appropriately defined before solving the valuation problem.

As an application of the BB lattice framework introduced in this paper we consider the pricing problem at time $t_0 \geq 0$ of European and American options written on a real estate asset whose value process X follows (2.1) and with payoff at time T (i.e. terminal condition) given by

$$
H(T) = \begin{cases} \Big(X(T)-K\Big)^+, & \text{for a call option} \\ \Big(K-X(T)\Big)^+, & \text{for a put option} \end{cases}
$$

where $(x)^+ = \max(x,0)$, $T \geq t_0$ is the maturity date and $K \geq 0$ is the strike price of the option. Let $V_E(n,i,j)$ and $V_A(n,i,j)$ denote the value at node (n,i,j), for $n = 0,\, 1,\, \ldots,\, N$ and $|i|,|j| \leq n$, of the European and American option, respectively. Using a calibrated BB tree with the life of the option $\tau := T - t_0$ divides into N equal time periods (or steps) of length $\Delta t = \tau/N$ years, the values of the European and American options at the maturity date T, i.e. at the N-th time period, are determined by the corresponding payoff as follows

$$
\Pi(N,i,j) \equiv H(N,i,j) = \begin{cases} \Big(X(N,i,j)-K\Big)^+, & \text{for a call option} \\ \Big(K-X(N,i,j)\Big)^+, & \text{for a put option} \end{cases}
\tag{4.2}
$$

where $X(N,i,j) = X(t_0)\exp\Big(i\Delta y(N,j)\Big)$, for $|i|,|j| \leq N$, are all of the possible values of the real estate asset at the final period N relative to the initial value $X(t_0)$ and to the possible realizations of the short rate $r(N,i,j)$ by which is determined the drift term for $\Delta y(N,j)$.

As we have shown above the value of the derivative at any node (n,i,j), with $n < N$, in the recombining tree is related to the four connecting nodes at the following time period $n+1$ according to the general discounted expectation formula (4.1). Specifically, for European options this backward induction procedure only has to be performed as far back as time period N when the terminal condition of the option (4.2) is implemented. Hence the European option value at node (n, i, j) is given by

$$V_E(n,i,j) = \Pi(n,i,j), \quad \forall\, i,j \text{ at period } n < N. \tag{4.3}$$

For American options, in order to evaluate the possibility of early exercise when applying the backward induction procedure by equation (4.1), we need to take the maximum of the discounted expectation and the intrinsic value (i.e. boundary conditions) of the option at each node. Similarly, after that the terminal condition of the option (4.2) has been implemented, the American option value at node (n, i, j), for all i,j at period $n < N$, is then given by

$$V_A(n,i,j) = \begin{cases} \Big[\Pi(n,i,j),\ \big(X(n,i,j) - K\big)\Big]^+, & \text{for a call option} \\ \Big[\Pi(n,i,j),\ \big(K - X(n,i,j)\big)\Big]^+, & \text{for a put option} \end{cases} \tag{4.4}$$

4.1 Numerical results and discussion

Once the branching process with both jump sizes and joint probabilities are correctly determined in such way that the resulting BB tree is consistent with market data, the general backward recursive procedure along with the appropriate terminal and boundary conditions allow us to value any specific contingent claim dependent upon the two state variables.

The numerical results reported in Table 1 show how the standard BSM model can be successfully recovered by implementing a BB lattice consistent with a flat yield curve and with yield volatilities and correlation coefficient set to be zero[2]. European and American options written on a real estate asset without any income flow, i.e. $\delta = 0$, are used for this purpose. The value at initial time $t_0 = 0$ of the underlying asset, $X(0)$, is 95, 100 or 110, the strike price, K, is 100, the time of maturity, T, is six months or one year, the instantaneous volatility of the percentage change in real estate asset value, σ_X, is of 20 or 30 per cent per annum and the risk-free interest rate, $r := r(t_0, T)$, is 5 per cent per annum. We assume that the two state variables are uncorrelated while the yield curve is flat at 5 per cent and the spot rate volatility is equal to zero, that is $\rho(t) \equiv 0$, $r(t) \equiv r$ and $\sigma_r(t) \equiv 0$, $\forall\, t \geq 0$. To implement the BB tree we use a number of time periods, N, that ranges from 20 to 240 while the analytic solution is calculated using the standard BSM pricing formula. The Crank-Nicolson finite difference (CNFD) method with a number of time steps equals to 1500 and 3000 for six-months and one-year options, respectively, is

[2] All algorithms are implemented in Matlab 6.5.

used to price the American put options since in the absence of income flows there is the usual equivalence between American and European calls.

[Table 1 about here]

From the results in Table 1, it is clear that the option prices obtained by the BB model converge with a slightly oscillatory behavior to both analytical and numerical solutions for any given value of the model parameters. Besides demonstrating numerical convergence to the BSM model and the CNFD method, an important analytical remark on the features of the two-factor pricing model when the spot interest rate is assumed constant may be useful to be considered. In fact in the special case of constant volatility structure, the two-factor pricing model is described by the following continuous-time risk-neutralized SDEs:

$$d\ln X(t) = \nu(t)dt \,+\, \sigma_X d\tilde{W}_1(t),$$

$$d\ln r(t) = \frac{\partial \ln u(t)}{\partial t}\, dt \,+\, \sigma_r\, d\tilde{W}_2(t).$$

where $\mathrm{corr}(d\tilde{W}_1(t), d\tilde{W}_2(t)) = \rho(t)$ and $u(t)$ is the median of the spot rate distribution at time $t \geq 0$.

When we assume that the two state variables X and r are uncorrelated and the initial yield curve is flat with the spot rate volatility equals zero, the diffusion process for r reduces to an ordinary differential equation of the form $d\ln r(t) = 0$. This result along with the initial condition $r(t_0) = r$ implies that the spot interest rate is a constant function, i.e.

$$r(t) = r, \quad \forall\, t \geq 0 \text{ and } r \in \mathbb{R}^+,$$

which gives the drift term for $d\ln X(t)$ to be constant, that is $\nu(t) \equiv \nu := (r - \delta - \frac{1}{2}\,\sigma_X^2)$. It follows that the jump sizes for $\ln r$ are equal to zero and then the BB lattice used to approximate the continuous-time diffusion processes reduces to a standard univariate binomial tree with time-invariant risk-neutral probabilities and additive jumps. More specifically, from the system consisting of equations (2.6)-(2.7) and (2.11)-(2.14) it turns out that the equal jump size and the unconditional probabilities of upward and downward jumps for $\ln X$ are respectively given by

$$\Delta y(t) = \sqrt{\sigma_X^2 \Delta t + \nu^2 \Delta t^2} := \Delta y, \tag{4.5}$$

and

$$p_{uu}(t) + p_{ud}(t) = \frac{1}{2} + \frac{1}{2}\frac{\nu\Delta t}{\Delta y} \;:=\; p_X, \tag{4.6}$$

$$p_{du}(t) + p_{dd}(t) = \frac{1}{2} - \frac{1}{2}\frac{\nu\Delta t}{\Delta y} \;=\; 1 - p_X \;:=\; q_X. \tag{4.7}$$

Note that in this particular formulation of the BB model we cannot obtain separately each of the joint probabilities, $p_{uu}(t)$, $p_{ud}(t)$, $p_{du}(t)$ and $p_{dd}(t)$, by solving the system of equations (2.6)-(2.7) and (2.11)-(2.14). Hence the BB lattice collapses into a standard binomial tree in which the equal jump size (4.5) and the risk-neutral probabilities (4.6)-(4.7) are identical to those of the binomial model originally developed by [5].

In order to assess the general validity of the BB model, we must take into consideration a stochastic interest rate and then fit the BB lattice to the initial yield and volatility curves. To simplify the analysis throughout this section, we assume that the volatility term structure is constant at level of 5 per cent per annum, i.e. $\sigma_r(t) \equiv 0.05\ \forall\, t \geq 0$, while the term structure of interest rates can be rising or declining according to the behavior described by the respective initial yield curve. Let us consider a time horizon T of one year and divide it into $N_T = 20$ periods, each having length $\Delta T := 1/N_T = 0.05$ years. Assuming that the yield on pure discount bond maturing at the end of the first time interval ΔT is equal to 5 per cent per annum, we build the BB tree consistent with the following two different interest rate term structures:

(a) *Increasing initial yield curve*: the interest rate increases from 5 per cent after ΔT years to 6 per cent after $N_T \cdot \Delta T = 1$ year and remains constant during each time interval of size ΔT;
(b) *Decreasing initial yield curve*: the interest rate decreases from 5 per cent after ΔT years to 4 per cent after $N_T \cdot \Delta T = 1$ year and remains constant during each time interval of size ΔT.

Since the surface of the option value obtained under the BB model is consistent with that calculated using the standard BSM pricing formula, we take a closer look at what are the patterns of the differences of European option values with stochastic interest rate minus the option values with constant interest rate. To calculate the surface of these differences over time to maturity and across different moneyness we use for common parameters of the two models the following values: $K = 100$, $r = 0.05$, $\delta = 0$, $\sigma_X = 0.2$, $\rho(t) \equiv 0$ $\forall\, t \geq 0$, with $X(t_0)$ ranges from 90 to 110, and time to maturity $\tau = \bar{T} - t_0$ from 0 to 1 year. To ensure that option prices with different maturities are broadly comparable, the number of periods N in which is divided the time horizon of the option varies such that the length Δt of each period is equal to 0.00625. Figures 3 and 4 show the various effects of stochastic interest rate in pricing European call and put options respectively, distinguishing between the two different initial yield curves that have been specified above. As we can see from Figures 3(a) and 4(a), when the term structure is rising, then the constant-rate model systematically underprices European calls and overprices European puts with respect to the stochastic-rate model. The degree of mispricing increases proportionately with the time to maturity and the absolute moneyness. The findings are just the opposite when the term structure is falling, as shown in Figures 3(b) and 4(b). The constant-rate model overprices European calls and underprices European puts with the mispricing that is largest for long time-to-maturity and in-the-money options.

[Figure 3 about here]

[Figure 4 about here]

When taking non-zero correlation coefficient into consideration, the magnitude of differences between option prices under stochastic-rate using the BB lattice and those calculated by the BSM pricing formula with constant-rate can exhibit some interesting patterns. Tables 2 and 3 report and compare prices and percentage pricing differences of European calls and puts respectively, under different constant levels of the correlation coefficient ρ and over several times of maturity and moneyness levels. As previously we use for common parameters the following values: $K = 100$, $\delta = 0$, $r = 0.05$ and $\sigma_X = 0.2$. The BB tree is fitted to the two different initial yield curves with constant spot rate volatility of 0.05 and using a number of periods N that varies such that the length Δt of each time step is equal to 0.005.

Table 2 shows that the BSM model underprices European calls when the initial yield curve has an upward slope and overprices European calls when the initial yield curve has a downward slope. As the time of maturity increases, the percentage pricing errors increases almost proportionately in absolute value. Table 2 reveals that, except for short-maturity options (three months), the absolute-percentage pricing error of out-of-the money call (i.e., $X_0 < K$) is largest compared to at-the-money and in-the-money calls (i.e., $X_0 \geq K$). Specifically, percentage differences for medium- and long-maturity call options (six months and one year) are decreasing functions of the moneyness level. Finally, comparison of percentage differences across the several ρ values reveals that the degree of mispricing increases as the correlation coefficient ρ ranges from -0.4 to 0.4 when the term structure is rising, while the pricing error decreases as ρ varies from negative to positive values when the term structure is falling.

[Table 2 about here]

[Table 3 about here]

Table 3 shows that for European puts the sign of the percentage pricing differences under stochastic- and constant-rate using respectively BB and BSM models is opposite to that of the European calls. Specifically, when the initial yield curve is upward sloping European puts turn out to be overvalued by the BSM model, while it undervalues European puts when the initial yield curve is downward sloping. Similarly, longer maturity implies larger effects of mispricing and, except for three-months options, as the moneyness level decreases the percentage differences in absolute terms increases. Finally, the effect of the correlation coefficient on the relative magnitude of pricing errors is the same as for European calls regarding to the direction of mispricing. Hence, the absolute-percentage differences decreases as ρ ranges from -0.4 to 0.4 when the option is underpriced (i.e. when the term structure is rising), while percentage differences increases as ρ varies from negative to positive values when the option is overpriced (i.e. when the term structure is falling).

Although we have shown graphically and numerically the effects of stochastic interest rate only on European call and put options, it is easy to verify that the obtained results are valid for both European and American options written on real estate assets without income flow or with constant continuously paid cash-flow.

5 Conclusions

In this paper, we develop a two-factor model that is both computationally efficient and numerically accurate for pricing the growing class of interest rate sensitive real estate derivatives. The kernel of the model is made of a spot rate process with drift and volatility terms consistent with the current market term structures and a possibly correlated underlying real estate value process. These diffusion processes are approximated in discrete-time by a bidimensional binomial (BB) model. An analytical solution for jumps measure and risk-neutral probabilities are derived with the attractive property to avoid the negative-probability problem. The calibration procedure to market data is based on the forward induction technique which involves using the Arrow-Debreu securities.

Numerical results show that option prices obtained under the BB model with constant spot rate and zero correlation converges rapidly to those calculated using the BSM pricing formula. Compared with the constant-rate model, the BB lattice framework turns out to be more accurate in pricing options for non-flat yield and volatility curves. In addition, the numerical tests show clear evidence supporting the use of the proposed model also when a low degree of correlation between the state variables is assumed.

References

[1] F. Black, M. Scholes, The pricing of options and corporate liabilities. *Journal of Political Economy* **81** 637-654 (1973).

[2] R. Merton, Theory of rational option pricing. *Bell Journal of Economics and Management Science* **4** 141-183 (1973).

[3] F. Black, E. Derman, W. Toy, A one-factor model of interest rates and its applications to treasury bond options. *Financial Analysts Journal* **46** 33-39 (1990).

[4] R. Buttimer, J. Kau, C. Slawson, A model for securities dependent upon a real estate index. *Journal of Housing Economics* **6** 16-30 (1997).

[5] L. Trigeorgis, A log-transformed binomial numerical analysis method for valuing complex multi-option investments. *Journal of Financial and Quantitative Analysis* **26** 309-326 (1991).

[6] J. Cox, S. Ross, M. Rubinstein, Option pricing: a simplified approach. *Journal of Financial Economics* **7** 229-263 (1979).

[7] F. Jamshidian, Forward induction and construction of yield curve diffusion models. *Journal of Fixed Income* **1** 62-74 (1991).

[8] K. Sandmann, D. Sondermann, A note on the stability of lognormal interest rate models and the pricing of Eurodollar futures. *Mathematical Finance* **7** 119-125 (1997).

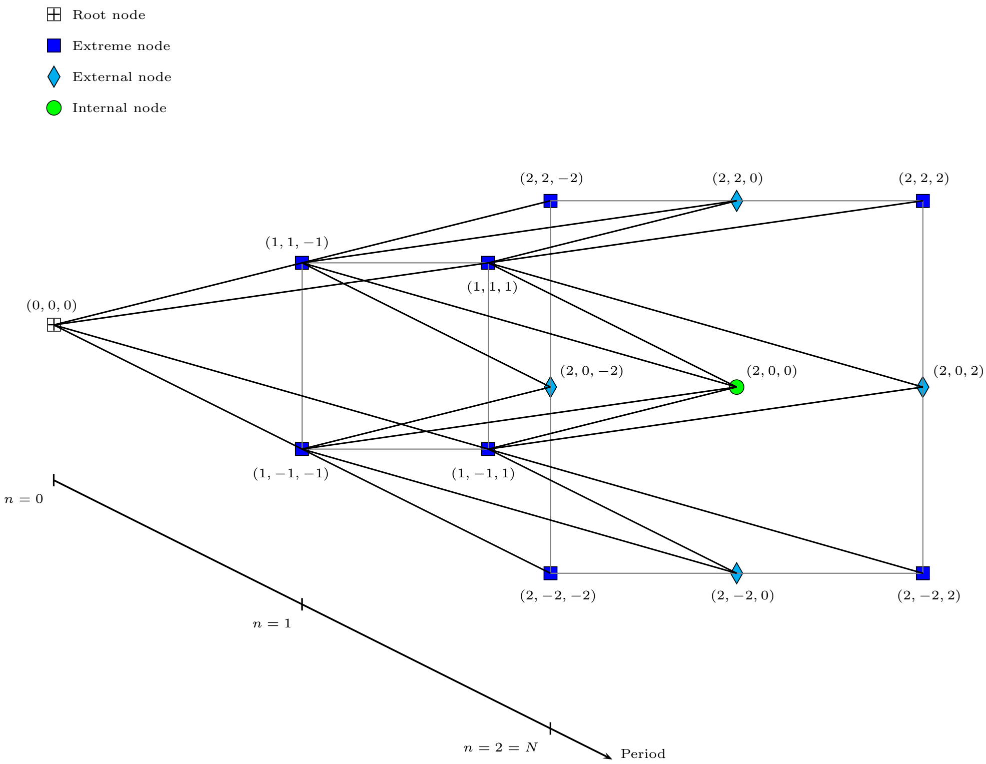


Fig. 1. Bidimensional binomial tree with $N = 2$ periods.

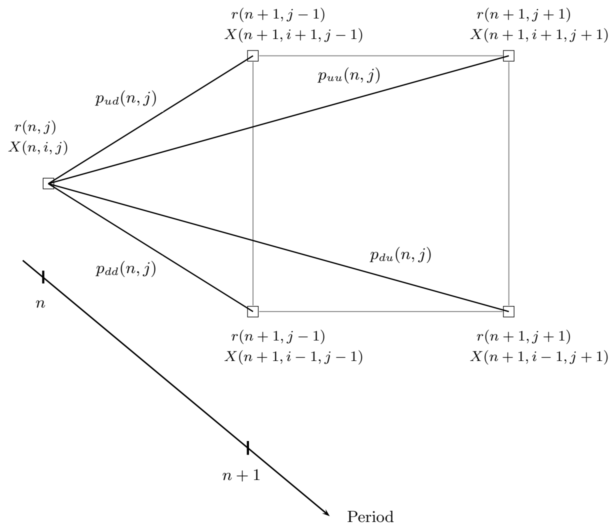
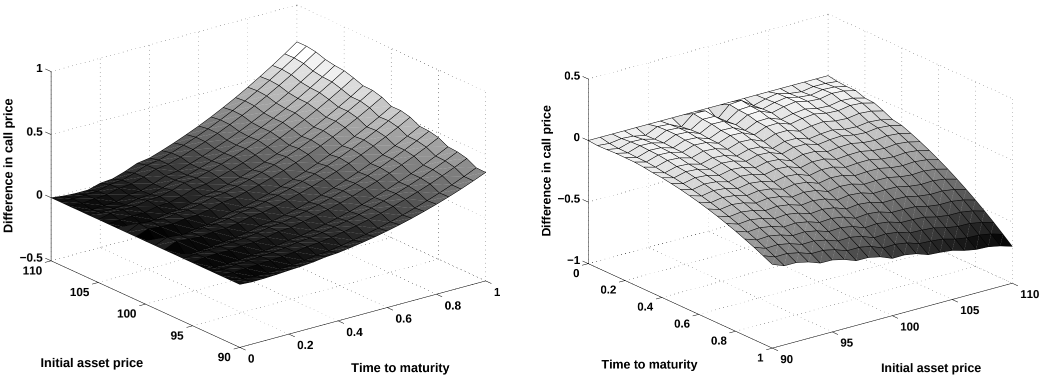


Fig. 2. Branching process for a bidimensional binomial tree

Table 1
Convergence to BSM model and CNFM method of European and American Vanilla Options under the BB model

No of periods	$T = 6$ months						$T = 1$ year					
	$\sigma_X = 0.2$			$\sigma_X = 0.3$			$\sigma_X = 0.2$			$\sigma_X = 0.3$		
	$X_0 = 95$	$X_0 = 100$	$X_0 = 105$	$X_0 = 95$	$X_0 = 100$	$X_0 = 105$	$X_0 = 95$	$X_0 = 100$	$X_0 = 105$	$X_0 = 95$	$X_0 = 100$	$X_0 = 105$
European call (American call)												
20	4.2642	6.8197	10.2295	7.0222	9.5300	12.8970	7.5923	10.3537	13.9438	11.3880	14.0824	17.6148
40	4.2508	6.8541	10.1893	6.9478	9.5823	12.8281	7.5169	10.4020	13.8767	11.3374	14.1566	17.5720
60	4.2728	6.8656	10.2166	6.9088	9.5998	12.7910	7.4780	10.4182	13.8407	11.3067	14.1814	17.5440
80	4.2676	6.8714	10.2164	6.9165	9.6085	12.7783	7.5086	10.4263	13.8495	11.2860	14.1939	17.5246
100	4.2562	6.8748	10.2088	6.9333	9.6138	12.7981	7.5207	10.4311	13.8636	11.2709	14.2013	17.5103
120	4.2435	6.8772	10.1991	6.9397	9.6173	12.8069	7.5239	10.4344	13.8688	11.2592	14.2063	17.4991
160	4.2591	6.8800	10.2024	6.9399	9.6217	12.8105	7.5201	10.4384	13.8685	11.2649	14.2125	17.4906
200	4.2603	6.8818	10.2071	6.9341	9.6243	12.8071	7.5120	10.4408	13.8631	11.2767	14.2163	17.5045
240	4.2553	6.8829	10.2052	6.9268	9.6261	12.8015	7.5031	10.4425	13.8563	11.2812	14.2188	17.5107
BSM formula	**4.2545**	**6.8887**	**10.2013**	**6.9282**	**9.6349**	**12.7986**	**7.5109**	**10.4506**	**13.8579**	**11.2733**	**14.2313**	**17.5051**
European put												
20	6.7959	4.3513	2.7612	9.5541	7.0620	5.4291	7.7178	5.4793	4.0695	11.5148	9.2095	7.7421
40	6.7821	4.3854	2.7206	9.4792	7.1138	5.3596	7.6411	5.5263	4.0011	11.4623	9.2816	7.6971
60	6.8040	4.3968	2.7479	9.4401	7.1311	5.3223	7.6018	5.5420	3.9646	11.4309	9.3057	7.6684
80	6.7988	4.4025	2.7476	9.4477	7.1398	5.3096	7.6321	5.5499	3.9732	11.4099	9.3178	7.6486
100	6.7873	4.4060	2.7399	9.4645	7.1450	5.3293	7.6441	5.5546	3.9871	11.3946	9.3251	7.6341
120	6.7746	4.4083	2.7302	9.4709	7.1485	5.3381	7.6473	5.5577	3.9922	11.3828	9.3299	7.6227
160	6.7902	4.4111	2.7335	9.4710	7.1528	5.3417	7.6434	5.5617	3.9918	11.3883	9.3360	7.6141
200	6.7914	4.4128	2.7382	9.4652	7.1554	5.3382	7.6351	5.5641	3.9863	11.4001	9.3396	7.6279
240	6.7864	4.4140	2.7362	9.4578	7.1572	5.3326	7.6262	5.5656	3.9794	11.4045	9.3421	7.6340
BSM formula	**6.7855**	**4.4197**	**2.7322**	**9.4592**	**7.1659**	**5.3295**	**7.6338**	**5.5735**	**3.9808**	**11.3963**	**9.3542**	**7.6280**
American put												
20	7.2289	4.6240	2.8942	9.8893	7.3361	5.5807	8.5044	6.0522	4.3813	12.2017	9.7992	8.1089
40	7.2320	4.6400	2.8459	9.8190	7.3656	5.5184	8.4544	6.0722	4.3305	12.1431	9.8351	8.0787
60	7.2409	4.6454	2.8688	9.7894	7.3749	5.4833	8.4398	6.0780	4.3008	12.1145	9.8470	8.0552
80	7.2347	4.6480	2.8700	9.7988	7.3799	5.4673	8.4562	6.0814	4.2998	12.0982	9.8529	8.0386
100	7.2262	4.6495	2.8641	9.8101	7.3828	5.4820	8.4611	6.0833	4.3094	12.0871	9.8562	8.0260
120	7.2194	4.6505	2.8561	9.8135	7.3847	5.4896	8.4617	6.0845	4.3136	12.0795	9.8585	8.0160
160	7.2290	4.6519	2.8565	9.8118	7.3870	5.4931	8.4577	6.0860	4.3142	12.0856	9.8616	8.0057
200	7.2287	4.6527	2.8606	9.8063	7.3884	5.4904	8.4525	6.0869	4.3105	12.0927	9.8633	8.0148
240	7.2250	4.6532	2.8593	9.8003	7.3894	5.4855	8.4483	6.0874	4.3055	12.0948	9.8645	8.0196
CNFD method	**7.2224**	**4.6539**	**2.8542**	**9.7990**	**7.3915**	**5.4793**	**8.4499**	**6.0891**	**4.3038**	**12.0852**	**9.8682**	**8.0125**

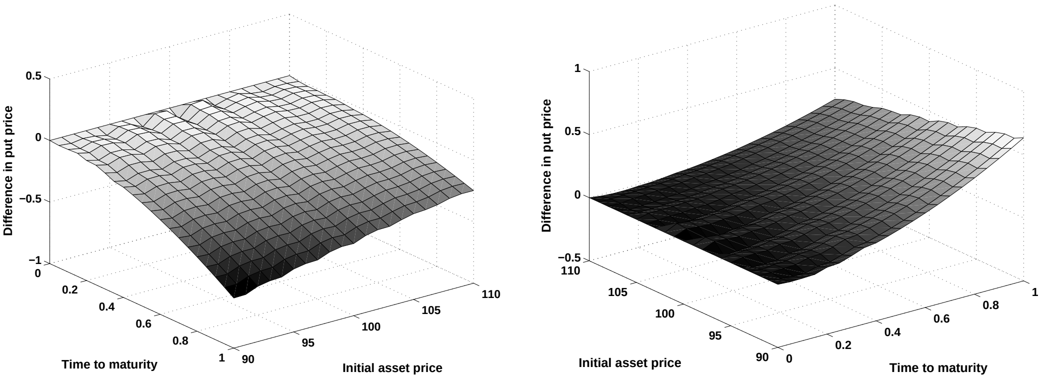
This table reports prices at initial time $t_0 = 0$ for European and American calls and puts under constant-rate assumption calculated using the BSM model and the BB model. The initial value of the underlying asset (X_0) is 95, 100 or 105, the strike price (K) is 100, the time of maturity (T) is 6 months or 1 year, the instantaneous volatility of the percentage change in real estate asset value (σ_X) is 0.2 or 0.3 per annum, the continuously paid cash-flow (δ) is 0, the risk-free interest rate (r) is 0.05 per annum, the yield curve is flat at 0.05, the instantaneous volatility of the percentage change in the spot rate ($\sigma_r(t)$) and the time-dependent correlation coefficient ($\rho(t)$) are equal to 0 $\forall\, t \geq 0$. The number of periods (N) used to implement the BB tree consistent with a flat initial yield curve at 5 cent ranges from 20 to 240 while the analytic solution is calculated using the standard BSM pricing formula. The CNFD method has a number of time steps equal to 1500 and 3000 for 6-months and 1-year options, respectively.



(a) Increasing initial yield curve (b) Decreasing initial yield curve

Fig. 3. Effects of stochastic interest rate in pricing European calls.

The left and right graphs of this figure show the pricing differences for European calls under stochastic-rate assumption using the BB model and within the standard BSM model with constant-rate, distinguishing between increasing and decreasing initial yield curves. To determine the surfaces, option prices are calculated for different moneyness levels choosing the initial value of the underlying asset (X_0) from 90 to 110 and for different times to maturity (τ) from 0 to 1 year. Values chosen for common parameters are $K = 100$, $r = 0.05$, $\delta = 0$, $\sigma_X = 0.2$, $\rho(t) \equiv 0$, $\forall\, t \geq 0$. The BB lattice is fitted to the two different initial yield curves with constant spot rate volatility (σ_r) of 0.05 and using a number of periods (N) that varies such that the length Δt of each time step is of 0.00625.



(a) Increasing initial yield curve (b) Decreasing initial yield curve

Fig. 4. Effects of stochastic interest rate in pricing European puts.

The left and right graphs of this figure show the pricing differences for European puts under stochastic-rate assumption using the BB model and within the standard BSM model with constant-rate, distinguishing between increasing and decreasing initial yield curves. To determine the surfaces, option prices are calculated for different moneyness levels choosing the initial value of the underlying asset (X_0) from 90 to 110 and for different times to maturity (τ) from 0 to 1 year. Values chosen for common parameters are $K = 100$, $r = 0.05$, $\delta = 0$, $\sigma_X = 0.2$, $\rho(t) \equiv 0$, $\forall\, t \geq 0$. The BB lattice is fitted to the two different initial yield curves with constant spot rate volatility (σ_r) of 0.05 and using a number of periods (N) that varies such that the length Δt of each time step is of 0.00625.

Table 2
Comparison between BSM and BB models for European calls with different ρ values

Panel A. Increasing initial yield curve

	$T=3/12$			$T=6/12$			$T=1$		
	$\rho=-0.4$	$\rho=0$	$\rho=0.4$	$\rho=-0.4$	$\rho=0$	$\rho=0.4$	$\rho=-0.4$	$\rho=0$	$\rho=0.4$
BB									
$X_0 = 90$	0.9058	0.9074	0.9091	2.4100	2.4164	2.4229	5.4272	5.4505	5.4739
$X_0 = 95$	2.2854	2.2877	2.2901	4.3441	4.3517	4.3594	7.9442	7.9688	7.9934
$X_0 = 100$	4.6153	4.6179	4.6205	6.9960	7.0039	7.0117	10.9711	10.9954	11.0196
$X_0 = 105$	7.9604	7.9626	7.9649	10.3610	10.3681	10.3752	14.4832	14.5058	14.5282
$X_0 = 110$	12.0281	12.0297	12.0313	14.2610	14.2669	14.2726	18.3676	18.3877	18.4076
Percentage Difference									
$X_0 = 90$	0.93%	1.10%	1.29%	2.58%	2.85%	3.13%	6.60%	7.06%	7.52%
$X_0 = 95$	0.62%	0.73%	0.83%	2.10%	2.28%	2.46%	5.77%	6.10%	6.42%
$X_0 = 100$	0.01%	0.06%	0.12%	1.56%	1.67%	1.79%	4.98%	5.21%	5.44%
$X_0 = 105$	0.47%	0.50%	0.53%	1.57%	1.64%	1.70%	4.51%	4.68%	4.84%
$X_0 = 110$	0.33%	0.35%	0.36%	1.32%	1.36%	1.40%	3.99%	4.10%	4.22%

Panel B. Decreasing initial yield curve

	$T=3/12$			$T=6/12$			$T=1$		
	$\rho=-0.4$	$\rho=0$	$\rho=0.4$	$\rho=-0.4$	$\rho=0$	$\rho=0.4$	$\rho=-0.4$	$\rho=0$	$\rho=0.4$
BB									
$X_0 = 90$	0.8921	0.8935	0.8950	2.2887	2.2935	2.2983	4.7571	4.7700	4.7830
$X_0 = 95$	2.2581	2.2602	2.2623	4.1590	4.1648	4.1705	7.0746	7.0886	7.1027
$X_0 = 100$	4.5722	4.5746	4.5769	6.7442	6.7502	6.7561	9.9058	9.9200	9.9342
$X_0 = 105$	7.9031	7.9051	7.9070	10.0482	10.0536	10.0590	13.2392	13.2526	13.2661
$X_0 = 110$	11.9606	11.9620	11.9634	13.8982	13.9027	13.9071	16.9679	16.9801	16.9922
Percentage Difference									
$X_0 = 90$	−0.60%	−0.44%	−0.28%	−2.59%	−2.38%	−2.18%	−6.56%	−6.31%	−6.05%
$X_0 = 95$	−0.58%	−0.48%	−0.39%	−2.24%	−2.11%	−1.97%	−5.81%	−5.62%	−5.43%
$X_0 = 100$	−0.93%	−0.88%	−0.83%	−2.10%	−2.01%	−1.92%	−5.21%	−5.08%	−4.94%
$X_0 = 105$	−0.25%	−0.23%	−0.20%	−1.50%	−1.45%	−1.39%	−4.46%	−4.37%	−4.27%
$X_0 = 110$	−0.23%	−0.22%	−0.21%	−1.26%	−1.23%	−1.20%	−3.93%	−3.87%	−3.80%

This table reports and compares prices at initial time $t_0 = 0$ for European calls under stochastic-rate assumption using the BB model and within the standard BSM model with constant-rate. To assess the general validity of the BB model, option prices are calculated for different moneyness levels choosing the initial value of the underlying asset (X_0) from 90 to 110, for different constant levels of the correlation coefficient setting $\rho(t) \equiv \rho \; \forall \, t \geq 0$, with ρ of -0.4, 0 and 0.4, and for different times of maturity (T) of 3 months, 6 months and 1 year. Values chosen for common parameters are $K = 100$, $\delta = 0$, $r = 0.05$ and $\sigma_X = 0.2$. The BB lattice is fitted to the two different initial yield curves with constant spot rate volatility (σ_r) of 0.05 and using a number of periods (N) that varies such that the length Δt of each time step is of 0.005.

Table 3
Comparison between BSM and BB models for European puts with different ρ values

Panel A. Increasing initial yield curve

	$T = 3/12$			$T = 6/12$			$T = 1$		
	$\rho = -0.4$	$\rho = 0$	$\rho = 0.4$	$\rho = -0.4$	$\rho = 0$	$\rho = 0.4$	$\rho = -0.4$	$\rho = 0$	$\rho = 0.4$
BB									
$X_0 = 90$	9.6241	9.6257	9.6272	9.7093	9.7155	9.7218	9.5967	9.6188	9.6410
$X_0 = 95$	6.0036	6.0059	6.0082	6.6434	6.6507	6.6582	7.1133	7.1367	7.1600
$X_0 = 100$	3.3335	3.3361	3.3386	4.2952	4.3028	4.3104	5.1397	5.1627	5.1856
$X_0 = 105$	1.6786	1.6808	1.6829	2.6602	2.6670	2.6738	3.6515	3.6727	3.6938
$X_0 = 110$	0.7464	0.7478	0.7493	1.5601	1.5657	1.5712	2.5355	2.5542	2.5726
Percentage Difference									
$X_0 = 90$	−0.32%	−0.31%	−0.29%	−1.73%	−1.67%	−1.61%	−6.05%	−5.83%	−5.61%
$X_0 = 95$	−0.42%	−0.38%	−0.35%	−2.10%	−1.99%	−1.88%	−6.82%	−6.51%	−6.21%
$X_0 = 100$	−1.16%	−1.09%	−1.01%	−2.82%	−2.64%	−2.47%	−7.78%	−7.37%	−6.96%
$X_0 = 105$	−0.12%	0.00%	0.13%	−2.64%	−2.39%	−2.14%	−8.27%	−7.74%	−7.21%
$X_0 = 110$	0.03%	0.23%	0.42%	−2.88%	−2.53%	−2.19%	−8.99%	−8.32%	−7.66%

Panel B. Decreasing initial yield curve

	$T = 3/12$			$T = 6/12$			$T = 1$		
	$\rho = -0.4$	$\rho = 0$	$\rho = 0.4$	$\rho = -0.4$	$\rho = 0$	$\rho = 0.4$	$\rho = -0.4$	$\rho = 0$	$\rho = 0.4$
BB									
$X_0 = 90$	9.6889	9.6903	9.6918	10.0510	10.0558	10.0606	10.8342	10.8468	10.8595
$X_0 = 95$	6.0549	6.0570	6.0592	6.9214	6.9271	6.9329	8.1515	8.1653	8.1790
$X_0 = 100$	3.3691	3.3714	3.3737	4.5066	4.5125	4.5185	5.9827	5.9966	6.0104
$X_0 = 105$	1.6999	1.7019	1.7039	2.8106	2.8160	2.8214	4.3159	4.3291	4.3421
$X_0 = 110$	0.7575	0.7589	0.7602	1.6606	1.6650	1.6695	3.0446	3.0564	3.0681
Percentage Difference									
$X_0 = 90$	0,35%	0.36%	0.38%	1.73%	1.78%	1.82%	6.07%	6.19%	6.32%
$X_0 = 95$	0,43%	0.46%	0.50%	2.00%	2.09%	2.17%	6.78%	6.96%	7.14%
$X_0 = 100$	−0,11%	−0,04%	0.03%	1.97%	2.10%	2.24%	7.34%	7.59%	7.84%
$X_0 = 105$	1,14%	1.26%	1.38%	2.87%	3.06%	3.26%	8.42%	8.75%	9.08%
$X_0 = 110$	1,53%	1.71%	1.89%	3.37%	3.65%	3.93%	9.29%	9.71%	10.13%

This table reports and compares prices at initial time $t_0 = 0$ for European puts under stochastic-rate assumption using the BB model and within the standard BSM model with constant-rate. To assess the general validity of the BB model, option prices are calculated for different moneyness levels choosing the initial value of the underlying asset (X_0) from 90 to 110, for different constant levels of the correlation coefficient setting $\rho(t) \equiv \rho \; \forall \, t \geq 0$, with ρ of -0.4, 0 and 0.4, and for different times of maturity (T) of 3 months, 6 months and 1 year. Values chosen for common parameters are $K = 100$, $\delta = 0$, $r = 0.05$ and $\sigma_X = 0.2$. The BB lattice is fitted to the two different initial yield curves with constant spot rate volatility (σ_r) of 0.05 and using a number of periods (N) that varies such that the length Δt of each time step is of 0.005.